ECE 601 Practice Problems Set #2 Solutions

1. Kailath Problem 2.4-3

(a) In the s-domain the equations of motion are:

$$s^2 I\,\theta(s) = mgl\,\theta(s) - s h\,\phi(s) \qquad (1)$$

$$s^2 J\,\phi(s) = s h\,\theta(s) + Q(s) \qquad (2)$$

From (1)

$$\frac{\theta(s)}{\phi(s)} = \frac{-sh}{s^2 I - mgl}.$$

From (2)

$$\boxed{\frac{\phi(s)}{Q(s)} = \frac{Is^2 - mgl}{IJs^4 + (h^2 - mglJ)s^2}.}$$

Since

$$\frac{\theta(s)}{Q(s)} = \frac{\theta(s)}{\phi(s)} \cdot \frac{\phi(s)}{Q(s)}$$

then

$$\boxed{\frac{\theta(s)}{Q(s)} = \frac{-h}{IJs^3 + (h^2 - mglJ)s}.}$$

(b) Define the state as $x = [\theta\ \dot{\theta}\ \phi\ \dot{\phi}]$. Then

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot\theta\\ \phi\\ \dot\phi\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{mgl}{I} & 0 & 0 & -\frac{h}{I}\\ 0 & 0 & 0 & 1\\ 0 & \frac{h}{J} & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta\\ \dot\theta\\ \phi\\ \dot\phi\end{bmatrix} + \underbrace{\begin{bmatrix}0\\0\\0\\ \frac{1}{J}\end{bmatrix}}_{b} Q$$

$$\mathcal{C}(b,A) = \begin{bmatrix} b & Ab & A^2 b & A^3 b\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\frac{h}{IJ} & 0\\ 0 & -\frac{h}{IJ} & 0 & \frac{h(h^2-mglJ)}{I^2J^2}\\ 0 & \frac{1}{J} & 0 & \frac{-h^2}{IJ^2}\\ \frac{1}{J} & 0 & -\frac{h^2}{IJ^2} & 0\end{bmatrix}$$

$$\det(\mathcal{C}(b,A)) = -\frac{h^2 mgl}{J^4 I^3} \neq 0$$

$\Rightarrow$ (A,b) is controllable.

Note $\phi(s)/Q(s)$ is irreducible and of degree 4 = dim (x). So using ϕ as an output gives observability. On the other hand, $\theta(s)/Q(s)$ is only of deg 3 < dim (x). So this output does not give observability.

(c) The poles/modes at $s=0$ imply instability.

2. Kailath Problem 2.4 – 14

Prove the contrapositive statement: If A and bc commute when $n \geq 2$ then (A, b, c) is NOT minimal. Observe that if A and bc commute then

$$\underbrace{(cAb)}_{\text{scalar}} b = b(cAb) \underset{\substack{\uparrow \\ \text{commutivity} \\ \text{used here}}}{=} A(cb)b = \underbrace{(cb)}_{\text{scalar}} Ab.$$

That is, the column vectors b and Ab are scalar multiples of each other. Hence, if $n \geq 2$, the first two columns of $\mathcal{C}(b, A)$ are not linearly independent. So the system is not controllable, and thus not minimal.

3. Kailath Problem 2.5-1

(a)

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad t \geq 0.$$

(b)

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad t \geq 0.$$

(c)

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & 1 & & \phi \\ & s & \ddots & \\ & & \ddots & 1 \\ \phi & & & s \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \cdots & \frac{1}{s^n} \\ & \frac{1}{s} & \ddots & \vdots \\ & \phi & & \frac{1}{s} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & \ddots & \vdots \\ \phi & & \ddots & 1 \end{bmatrix}, \quad t \geq 0.$$

(Toeplitz matrix)

(d)

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s-\lambda & 1 & & \phi \\ & s-\lambda & \ddots & \\ & & \ddots & 1 \\ \phi & & & s-\lambda \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s-\lambda} & \frac{1}{(s-\lambda)^2} & \cdots & \frac{1}{(s-\lambda)^n} \\ & \frac{1}{s-\lambda} & \ddots & \vdots \\ \phi & & & \frac{1}{s-\lambda} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!} e^{\lambda t} \\ & e^{\lambda t} & \ddots & \vdots \\ \phi & & \ddots & e^{\lambda t} \end{bmatrix}, \quad t \geq 0$$

(Toeplitz matrix)

(e) $$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots + \frac{(At)^{k-1}}{(k-1)!}, \quad t \geq 0$$

(f) $$e^{At} = \mathcal{L}^{-1}\left\{\left[\mathrm{diag}(sI-A_1, sI-A_2, \ldots, sI-A_N)\right]^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\mathrm{diag}\left((sI-A_1)^{-1}, (sI-A_2)^{-1}, \ldots, (sI-A_N)^{-1}\right)\right\}$$

$$= \mathrm{diag}\left(\mathcal{L}^{-1}\{(sI-A_1)^{-1}\}, \mathcal{L}\{(sI-A_2)^{-1}\}, \ldots \mathcal{L}^{-1}\{(sI-A_N)^{-1}\}\right)$$

$$= \mathrm{diag}\left(e^{A_1 t}, e^{A_2 t}, \ldots, e^{A_N t}\right), \quad t \geq 0$$

4. Kailath Problem 2.5-12

Using the hint with $s = k\Delta$ and $t = (k+1)\Delta$:

$$x((k+1)\Delta) = e^{A\Delta} x(k\Delta) + \int_{k\Delta}^{(k+1)\Delta} e^{A((k+1)\Delta - \tau)}\, b\, u_k \, d\tau .$$

With the change of variables $\bar{\tau} = (k+1)\Delta - \tau$

$$x((k+1)\Delta) = \underbrace{e^{A\Delta}}_{\Phi} x(k\Delta) + \underbrace{\int_0^{\Delta} e^{A\bar{\tau}}\, b\, d\bar{\tau}}_{\Gamma} \cdot u_k .$$

5. (a) Let $A = P\Lambda P^{-1}$ be the spectral decomposition of A. Observe

$$
\begin{aligned}
b(A) &= b_1 A^n + b_2 A^{n-1} + \cdots + b_n I \\
&= b_1 (P\Lambda P^{-1})^n + b_2 (P\Lambda P^{-1})^{n-1} + \cdots + b_n I \\
&= P\left[b_1 \Lambda^n + b_2 \Lambda^{n-1} + \cdots + b_n I \right] P^{-1} \\
&= P \operatorname{diag}\big(b(\lambda_1(A)), b(\lambda_2(A)), \ldots, b(\lambda_n(A)) \big) P^{-1}
\end{aligned}
$$

is the spectral decomposition of $b(A)$. Thus, $\lambda(b(A)) = b(\lambda(A))$.

(b) From the Cayley-Hamilton Theorem $b(A) = 0$. Thus,

$$\lambda(b(A)) = b(\lambda(A)) = 0.$$