ECE 601 Homework #6 Solutions

1. Kailath Problem 2.3-14

Observe

$$\mathcal{C}\left(\begin{bmatrix} b \\ 0 \end{bmatrix}, \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} b & Ab & \cdots & A^n b \\ 0 & cb & & cA^{n-1} b \end{bmatrix}$$

$$= \begin{bmatrix} A & b \\ c & 0 \end{bmatrix} \left[\begin{array}{c|c} 0 & \mathcal{C}(b, A) \\ \hline 1 & 0 \end{array}\right].$$

So the given pair is controllable if and only if both matrices on the right hand-side have full rank. Note the second matrix on the right has full rank if and only $\mathcal{C}(b, A)$ has full rank, i.e., (A, b) is controllable. This proves the claim in question.

2. Kailath Problem 2.3-26

(a) Assuming $y_k = y_{k-1} + y_{k-2}$, $k \geq 2$ then

$$\sum_{k=2}^{\infty} y_k z^{-k} = \sum_{k=2}^{\infty} (y_{k-1} + y_{k-2}) z^{-k}$$

$$y_0 + y_1 z^{-1} + \sum_{k=2}^{\infty} y_k z^{-k} = y_0 + y_1 z^{-1} + \underbrace{\sum_{k=0}^{\infty} y_{k-1} z^{-k} + \sum_{k=0}^{\infty} y_{k-2} z^{-k}}$$

since $y_0 = 0$ and defining $y_{-1} = y_{-2} = 0$

$$Y(z) = z^{-1} + z^{-1} Y(z) + z^{-2} Y(z)$$

$$Y(z) = \frac{z}{z^2 - z - 1} = \frac{\frac{1}{\sqrt{5}}\lambda_+}{z - \lambda_+} - \frac{\frac{1}{\sqrt{5}}\lambda_-}{z - \lambda_-}$$

$$= \frac{z^{-1}}{\sqrt{5}} \left(\lambda_+ \frac{z}{z - \lambda_+} - \lambda_- \frac{z}{z - \lambda_-} \right)$$

$$Y(k) = \frac{\lambda_+}{\sqrt{5}} \lambda_+^{k-1} - \frac{\lambda_-}{\sqrt{5}} \lambda_-^{k-1}$$

$$= \frac{1}{\sqrt{5}} \left(\lambda_+^k - \lambda_-^k \right), \quad k \geq 0. \checkmark$$

(b) Since $|\lambda_-| = 0.6180 < 1$

$$\lim_{n \to \infty} \frac{\ln y_n}{n} = \lim_{n \to \infty} \frac{\ln\left(\frac{1}{\sqrt{5}}(\lambda_+^n - \lambda_-^n)\right)}{n}$$

$$= \lim_{n \to \infty} \frac{\ln\left(\frac{1}{\sqrt{5}} \lambda_+^n\right)}{n}$$

$$= \ln \lambda_+ \checkmark$$

3. Kailath Problem 2.3-29

(a) In the z-domain with $\xi_0 = 0$

$$z\,\xi(z) = F\,\xi(z) + g\,z\,U(z)$$
$$Y(z) = h\,\xi(z).$$

Eliminating $\xi(z)$ gives $y(z) = h(zI-F)^{-1} g\,z\,U(z)$ so that the conclusion follows.

(b) $\xi(1) = F\,\xi(0) + g u(1)$

$$\xi(2) = F\xi(1) + gu(2) = F^2\xi(0) + Fg\,u(1) + gu(2)$$
$$\vdots$$
$$\xi(n) = F^n\xi(0) + \underbrace{\left[g \;\; Fg \cdots F^{n-1}g\right]}_{R(g,F)} \underbrace{\begin{bmatrix} u(n) \\ \vdots \\ u(1) \end{bmatrix}}_{\bar{u}(n)}$$

So given any $\xi(0)$ and $\xi(n)$, there exists a suitable $\bar{u}(n)$ if and only if $\det(R(g,F)) \neq 0$.

(c) Define $u(k+1) = \theta(k)$ s.t.

$$\begin{bmatrix} \xi(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} g \\ 1 \end{bmatrix} \theta(k)$$

$$y(k) = h\,\xi(k).$$

This looks like the conventional realization. To reproduce the result from part (a), for example, observe that

$$Y(z)/\theta(k) = h(zI-F)^{-1}g, \quad U(z)/\theta(z) = 1/z.$$

Combining gives

$$Y(z)/U(z) = z\,h(zI-F)^{-1}g.$$

4. See the attached MATLAB code. The output is shown below.

(a) The controllability matrix is:

```
C =

         0  322.4903

  322.4903         0
```

The matrix has full rank, so the system is controllable.

(b) The observability matrix is:

```
O =

     1     0

     0     1
```

The matrix has full rank, so the system is observable.

Remark: The results are expected since the transfer function is irreducible.

```
%
% ECE 601 Fall 2025
%
% Homework 6, Problem 4
%
% MATLAB R2023a
%

clc;
clear all;
close all;

% system parameters

m=0.01;
g=9.8;
ks=15;
ds=0.10;
km=0.005;
dm=0.12;

% equilibrium

x1e=0.11;
x2e=0;
xe=[x1e x2e]';
ue=sqrt(((ks/m)*(x1e-ds)-g)*(m/km)*(x1e-dm)^2);

% linear state space model

alpha=-(ks/m)-(2*km/m)*(((ue^2)/((x1e-dm)^3)));
beta=(2*km/m)*(ue/((x1e-dm)^2));
A=[0 1;alpha 0];
b=[0 beta]';
c=[1 0];
d=zeros(1,1);

% controllability and observability matrices

C=ctrb(A,b)
svC=svd(C)
O=obsv(A,c)
svO=svd(O)
```