ECE 302 Practice Problems Set #2 Solutions

Problem #01: Linear Transformation.

(a) $A = \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y+1, x-z, 2z)$

let's assume, $v_1 = (x_1, y_1, z_1)$ and
$v_2 = (x_2, y_2, z_2)$, $v_1, v_2 \in \mathbb{R}^3$

Superposition holds if:

$$A(a_1 v_1 + a_2 v_2) = a_1 A(v_1) + a_2 A_2(v_2),$$
$$a_1, a_2 \in R$$

take left side:

$$A(a_1 v_1 + a_2 v_2) = A\big((a_1 x_1, a_1 y_1, a_1 z_1) + (a_2 x_2, a_2 y_2, a_2 z_2)\big)$$

$$= A\big((a_1 x_1 + a_2 x_2), (a_1 y_1 + a_2 y_2), (a_1 z_1 + a_2 z_2)\big)$$

$$= \Big(\big((a_1 y_1 + a_2 y_2) + 1\big), \big((a_1 x_1 + a_2 x_2) - (a_1 z_1 + a_2 z_2)\big), \big(2(a_1 z_1 + a_2 z_2)\big)\Big)$$

$$A(a_1 v_1 + a_2 v_2) = \Big((a_1 y_1 + a_2 y_2 + 1), (a_1 x_1 + a_2 x_2 - a_1 z_1 - a_2 z_2), (2a_1 z_1 + 2a_2 z_2)\Big)$$

Now, take Right side:

$$a_1 A(V_1) + a_2 A_2(V_2) = a_1\left(y_1+1,\ x_1-z_1,\ 2z_1\right)$$
$$+ a_2\left(y_2+1,\ x_2-z_2,\ 2z_2\right)$$
$$= \left(a_1y_1+a_1,\ a_1x_1-a_1z_1,\ 2a_1z_1\right) +$$
$$\left(a_2y_2+a_2,\ a_2x_2-a_2z_2,\ 2a_2z_2\right)$$

$$a_1 A(V_1) + a_2 A_2(V_2) = \Big((a_1y_1+a_2y_2+a_1+a_2),$$
$$(a_1x_1+a_2x_2-a_1z_1-a_2z_2),\ (2a_1z_1+2a_2z_2)\Big)$$

as, a_1+a_2 not necessarily equal to 1, $a_1, a_2 \in R$
So, left side and right side are not equal and so Superposition doesn't hold for the mapping.

(b) $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 : (x, y) \longmapsto (x, y, y)$

Let's assume, $V_1 = (x_1, y_1)$ and
$V_2 = (x_2, y_2)$, $V_1, V_2 \in \mathbb{R}^2$

Superposition holds if:

$$A(a_1V_1 + a_2V_2) = a_1 A(V_1) + a_2 A_2(V_2),$$
$$a_1, a_2 \in R$$

take left side:

$$A(a_1V_1 + a_2V_2) = A\Big((a_1x_1, a_1y_1) + (a_2x_2, a_2y_2)\Big)$$

$$= A\Big((a_1x_1 + a_2x_2), (a_1y_1 + a_2y_2)\Big)$$

$$= \Big((a_1x_1 + a_2x_2), (a_1y_1 + a_2y_2), (a_1y_1 + a_2y_2)\Big)$$

$$= \Big((a_1x_1, a_1y_1, a_1y_1) + (a_2x_2, a_2y_2, a_2y_2)\Big)$$

$$= \Big(a_1(x_1, y_1, y_1) + a_2(x_2, y_2, y_2)\Big)$$

$$= a_1 A(V_1) + a_2 A(V_2) \rightarrow \text{Right side}$$

Since, left side equal to right side,

Hence,

superposition holds for the mapping $A: \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto (x, y, y)$.

Problem #2: Matrix representation:

(a) $A \cdot \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y+1, x-z, 2z)$

Since mapping A here is not linear as, $A(0) \neq 0$ so, there is no matrix representation for this mapping.

Note: Matrix representation is linear transformation always.

(b) $A : \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto (x, y, y)$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} x \\ y \\ y \end{bmatrix}}$$

Problem # 03: Composing Linear Transformations:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

(a) $A \circ B = ?$ Note: A and B are linear transformations in matrix forms, so $A \circ B = AB$ (composition of A and B equal to matrix multiplication)

So,

$$A \circ B = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2-2 & 6-2 \\ -1 & -3 \end{bmatrix}$$

$$A \circ B = \begin{bmatrix} 0 & 4 \\ -1 & -3 \end{bmatrix}$$

(b) $(A \circ B)^{-1} = ?$

$$(A \circ B)^{-1} = \begin{bmatrix} 0 & 4 \\ -1 & -3 \end{bmatrix}^{-1} = \frac{1}{0-(-4)} \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix}$$

$$(A \circ B)^{-1} = \frac{1}{4} \begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix}$$

(C) $B^{-1} \circ A^{-1} = ?$

$$B^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}^{-1} = \frac{1}{2-6}\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \frac{1}{0-1}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

So, $$B^{-1} \circ A^{-1} = \left(-\frac{1}{4}\right)(-1)\begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$B^{-1} \circ A^{-1} = \frac{1}{4}\begin{bmatrix} -3 & -4 \\ 1 & 0 \end{bmatrix}$$

Observation: $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$

Problem # 04: $A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

Sol

Eigen values of A:

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)(-\lambda) - (-1)(-1)$$
$$= (-2\lambda + \lambda^2) - 1$$
$$|A - \lambda I| = \lambda^2 - 2\lambda - 1 \text{ , for } |A - \lambda I| = 0$$

we get, $\lambda = 1 \pm \sqrt{2}$ → eigen values of A.

<u>eigen values of B:</u>

$$B - \lambda I = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}$$

$$|B - \lambda I| = (1-\lambda)(2-\lambda) - (2)(3)$$
$$= 2 - 2\lambda - \lambda + \lambda^2 - 6$$
$$|B - \lambda I| = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

So,

$\lambda = 4, -1$ → eigen values of B.

<u>Eigenvectors of A:</u> $A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ -1 & -\lambda \end{bmatrix}$

for $\lambda_1 = 1 + \sqrt{2}$,

$$A - \lambda_1 I = \begin{bmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{bmatrix}$$

So

$$(A - \lambda_1 I) V_1 = 0$$

$$\begin{bmatrix} 1-\sqrt{2} & -1 \\ -1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0 \text{ gives,}$$

$$-V_{11} + (-1-\sqrt{2}) V_{12} = 0 \Rightarrow (-1-\sqrt{2}) V_{12} = V_{11}$$

suggests,

$$V_{11} = -1-\sqrt{2} \quad \text{and} \quad V_{12} = 1$$

So,

$$V_1 = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \text{ is eigenvector of } A \text{ for } \lambda_1 = 1+\sqrt{2}$$

Similarly,

for $\lambda_2 = 1-\sqrt{2}$, $B-\lambda_2 I = \begin{bmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{bmatrix}$

and so, $(B-\lambda_2 I)V_2 = 0$

$$\begin{bmatrix} 1+\sqrt{2} & -1 \\ -1 & -1+\sqrt{2} \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives,}$$

$$-V_{21} + (-1+\sqrt{2})V_{22} = 0 \Rightarrow V_{21} = (-1+\sqrt{2})V_{22}$$

suggests, $V_{22} = 1$ and $V_{21} = \sqrt{2}-1$

So,

$$V_2 = \begin{bmatrix} \sqrt{2}-1 \\ 1 \end{bmatrix} \text{ is eigenvector of } A \text{ for } \lambda_2 = 1-\sqrt{2}.$$

<u>Eigenvectors of B:</u>

Since, $B-\lambda I = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}$

for $\lambda_1 = 4$, $B-\lambda_1 I = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix}$

and $(B-\lambda_1 I)V_1 = 0$

so, $\begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives,

$$-3V_{11} + 3V_{12} = 0 \Rightarrow -3V_{11} = -3V_{12}$$

Suggests, $V_{11} = 1 = V_{12}$

So, $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector of B for $\lambda_1 = 4$

Similarly for $\lambda_2 = -1$, $B - \lambda_2 I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

So, $(B - \lambda_2 I) V_2 = 0$

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow 2V_{21} + 3V_{22} = 0$$
$$\Rightarrow 2V_{21} = -3V_{22}$$

Suggests, $V_{21} = -3$ and $V_{22} = 2$

soo, $V_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is eigenvector of B for $\lambda_2 = -1$.

Problem #05: Matrix Exponential:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

(a) $e^{At} = ?$ Since $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$

so,

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (s-2) & 1 \\ 1 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s-2) - 1} \begin{bmatrix} s & -1 \\ -1 & s-2 \end{bmatrix}$$

$$(s\underline{I}-A)^{-1} = \frac{1}{s^2-2s-1}\begin{bmatrix} s & -1 \\ -1 & s-2 \end{bmatrix}$$

From s^2-2s-1 we get $s = 1 \pm \sqrt{2}$

So, $s^2-2s-1 = (s-1-\sqrt{2})(s-1+\sqrt{2})$

So $$(sI-A)^{-1} = \frac{1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\begin{bmatrix} s & -1 \\ -1 & s-2 \end{bmatrix}$$

$$(s\underline{I}-A)^{-1} = \begin{bmatrix} \frac{s}{(s-1-\sqrt{2})(s-1+\sqrt{2})} & \frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})} \\ \frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})} & \frac{(s-2)}{(s-1-\sqrt{2})(s-1+\sqrt{2})} \end{bmatrix}$$

So, $e^{At} = \mathcal{L}^{-1}\left((sI-A)^{-1}\right)$

So, if $e^{At} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then,

$$a_{11} = \mathcal{L}^{-1}\left(\frac{s}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right)$$

$$a_{11} = \frac{(1+\sqrt{2})e^{(1+\sqrt{2})t} + (\sqrt{2}-1)e^{(1-\sqrt{2})t}}{2\sqrt{2}}$$

and

$$a_{12} = \mathcal{L}^{-1}\left(\frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right)$$

$$a_{12} = \frac{-e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}}{2\sqrt{2}}$$

Note: $a_{21} = a_{12} = \dfrac{-e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}}{2\sqrt{2}}$

and finally,

$$a_{22} = \mathcal{L}^{-1}\left(\frac{(s-2)}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right)$$

$$a_{22} = \frac{(\sqrt{2}-1)e^{(1+\sqrt{2})t} + (1+\sqrt{2})e^{(1-\sqrt{2})t}}{2\sqrt{2}}$$

<u>Inverse Laplace Calculations Basics:</u>

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2-2s-1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2-2}\right)$$

Note: if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

Above, we have $s-a = s-1 \Rightarrow a=1$

so $\mathcal{L}^{-1}\left(\dfrac{1}{(s-1)^2-2}\right) = e^t \mathcal{L}^{-1}\left(\dfrac{1}{s^2-a}\right)$

Now, $\mathcal{L}^{-1}\left(\frac{1}{s^2-a}\right) = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{s^2-(\sqrt{a})^2}\right)$

$\because \mathcal{L}^{-1}(a f(s)) = a\mathcal{L}^{-1}(f(s))$

So, $\mathcal{L}^{-1}\left(\frac{1}{s^2-2}\right) = \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right)$

$\because \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh(at)$

So,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-(\sqrt{2})^2}\right) = \frac{1}{\sqrt{2}} \sinh(\sqrt{2}t)$$

So, $\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2-2}\right) = e^t \frac{1}{\sqrt{2}} \sinh(\sqrt{2})t$

or

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right) = e^t \frac{1}{\sqrt{2}} \sinh(\sqrt{2})t$$

and so,

$$\mathcal{L}^{-1}\left(\frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right) = -e^t \frac{1}{\sqrt{2}} \sinh(\sqrt{2})t$$

$\because \sinh(\sqrt{2})t = \frac{e^{\sqrt{2}t} - e^{-\sqrt{2}t}}{2}$

So,

$$\mathcal{L}^{-1}\left(\frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right) = -\frac{e^t\left(e^{\sqrt{2}t} - e^{-\sqrt{2}t}\right)}{2\sqrt{2}}$$

$$\mathcal{L}^{-1}\left(\frac{-1}{(s-1-\sqrt{2})(s-1+\sqrt{2})}\right) = \frac{-e^{(1+\sqrt{2})t} + e^{(1-\sqrt{2})t}}{2\sqrt{2}}$$

that's how $a_{12} = a_{21}$ is calculated.

in similar way you can calculate a_{11} and a_{22} to get the matrix form of e^{At}.

Pb#05 (b) and (c)

Eigen values and Eigenvectors of e^{At}:

transformation matrix $P = \begin{bmatrix} -1-\sqrt{2} & \sqrt{2}-1 \\ 1 & 1 \end{bmatrix}$

matrix containing eigenvectors of A.

$D = \begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}$ diagonal matrix with eigenvalues of A.

so, $A = PDP^{-1}$

taylor series expansion of e^{At}:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

Substitute $A = PDP^{-1}$

$$e^{At} = I + (PDP^{-1})t + \frac{(PDP^{-1})(PDP^{-1})t^2}{2!} + \dots$$

$$e^{At} = P\left(I + Dt + \frac{D^2t^2}{2!} + \cdots\right)P^{-1}$$

So, $$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{(1+\sqrt{2})t} & 0 \\ 0 & e^{(1-\sqrt{2})t} \end{bmatrix}P^{-1}$$

So, eigen values of e^{At} are $e^{(1+\sqrt{2})t}$ and $e^{(1-\sqrt{2})t}$

and eigen vectors are same the coloumns of P matrix

that are $\begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \sqrt{2}-1 \\ 1 \end{bmatrix}$