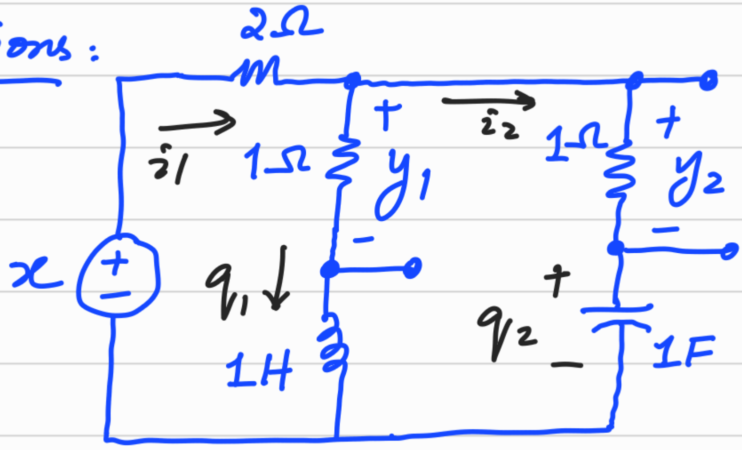
ECE 302 – HW#08 SOLUTIONS FALL 2025

or 10.3-4 in some textbook versions

Pb#01: (10.2-4) (two outputs y_1 and y_2)

State and output equations:



Loop equations:

$$\Rightarrow \quad i_2 = \frac{dq_2}{dt} = \dot{q}_2$$

$$\Rightarrow \quad i_1 = q_1 + i_2 = q_1 + \dot{q}_2$$

so,

$$x = 2i_1 + q_1 + \dot{q}_1 = 2(q_1 + \dot{q}_2) + q_1 + \dot{q}_1$$

$$x = 3q_1 + \dot{q}_1 + 2\dot{q}_2 \quad \rightarrow (1)$$

And,

$$x = 2i_1 + \dot{q}_2 + q_2 = 2(q_1 + \dot{q}_2) + \dot{q}_2 + q_2$$

$$x = 2q_1 + q_2 + 3\dot{q}_2 \quad \rightarrow (2)$$

Rearrange eq (2), to get,

$$\Rightarrow \quad \dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x \quad ,$$ substitute in eq (1) to get,

$$\Rightarrow \quad \dot{q}_1 = -\frac{5}{3}q_1 + \frac{2}{3}q_2 + \frac{1}{3}x$$

So, the state equations are:

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}-5/3 & 2/3\\ -2/3 & -1/3\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix}1/3\\ 1/3\end{bmatrix}x(t)$$

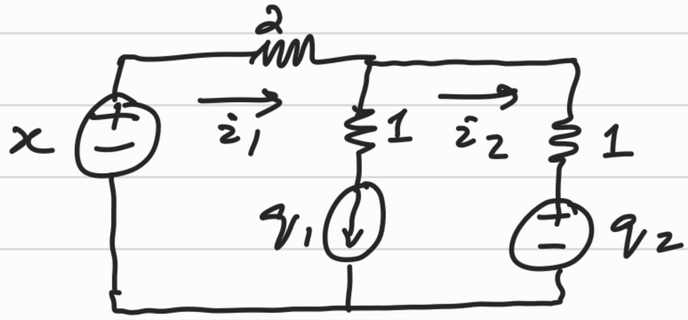
output equations are $y_1 = q_1$ and $y_2 = i_2 = \dot{q}_2$

so, $y_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x$

so,

$$y = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ -2/3 & -1/3\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix}0\\ 1/3\end{bmatrix}x(t)$$

equivalent circuit:



Pb#02: Input-Output System:

(a) $\frac{Y_1(s)}{X(s)}$, $\frac{Y_2(s)}{X(s)}$

Since, $y_1 = \underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{C_1}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \overbrace{[0]}^{D_1}x(t)$

And,

$$A = \begin{bmatrix}-5/3 & 2/3\\ -2/3 & -1/3\end{bmatrix}, \quad B = \begin{bmatrix}1/3\\ 1/3\end{bmatrix}$$

so, $\frac{Y_1(s)}{X(s)} = C_1(sI-A)^{-1}B + \overbrace{D_1}^{=0} = C_1(sI-A)^{-1}B$

So, $(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -5/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} s+5/3 & -2/3 \\ 2/3 & s+1/3 \end{bmatrix}$

And $(sI - A)^{-1} = \dfrac{Adj(sI-A)}{|sI-A|}$

$$(sI-A)^{-1} = \frac{1}{\left(s+\frac{5}{3}\right)\left(s+\frac{1}{3}\right)+\frac{4}{9}} \begin{bmatrix} (s+1/3) & 2/3 \\ -2/3 & (s+5/3) \end{bmatrix}$$

$$= \frac{1}{s^2+\frac{1}{3}s+\frac{5}{3}s+\frac{5}{9}+\frac{4}{9}} \begin{bmatrix} \left(\frac{3s+1}{3}\right) & 2/3 \\ -2/3 & \left(\frac{3s+5}{3}\right) \end{bmatrix}$$

$$= \frac{1}{s^2+\frac{1}{3}s+\frac{5s}{3}+1} \begin{bmatrix} \left(\frac{3s+1}{3}\right) & 2/3 \\ -2/3 & \left(\frac{3s+5}{3}\right) \end{bmatrix}$$

$$= \frac{3}{3s^2+6s+3} \begin{bmatrix} \left(\frac{3s+1}{3}\right) & 2/3 \\ -2/3 & \left(\frac{3s+5}{3}\right) \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{3s^2+6s+3} \begin{bmatrix} (3s+1) & 2 \\ -2 & (3s+5) \end{bmatrix}$$

$$(sI-A)^{-1}B = \frac{1}{3s^2+6s+3} \begin{bmatrix} \left(\frac{3s+1}{3}\right)+\frac{2}{3} \\ -\frac{2}{3}+\left(\frac{3s+5}{3}\right) \end{bmatrix}_{2\times 1}$$

And,

$$C_1(sI-A)^{-1}B = \frac{1}{3s^2+6s+3}\left[\left(\frac{3s+1}{3}\right)+\frac{2}{3}\right]_{1\times 1}$$

$$= \frac{1}{3s^2+6s+3}\left(\frac{3s+3}{3}\right)$$

$$C_1(sI-A)^{-1}B = \boxed{\frac{(s+1)}{(3s^2+6s+3)} = \frac{Y_1(s)}{X(s)}}$$

From MATLAB:

Define A, B, C, D

then, G = ss(A, B, C, D)

tf(G)

gives,

$$G(s) = \frac{0.333s + 0.333}{s^2+2s+1} = \frac{1/3\,(s+1)}{s^2+2s+1}$$

$$\boxed{G(s) = \frac{(s+1)}{3s^2+6s+3}}$$ → exact what we got by calculation.

Now, similar calculations will be for

$$\frac{Y_2(s)}{X(s)} = C_2(sI-A)^{-1}B + D_2$$

where,

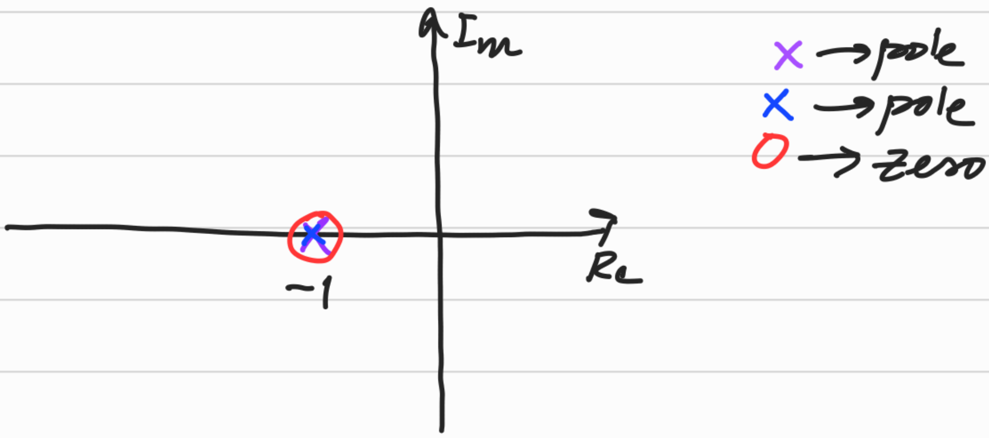
$$C_2 = [-2/3 \quad -1/3] \text{ and } D_2 = [1/3]$$

So, from MATLAB, we got,

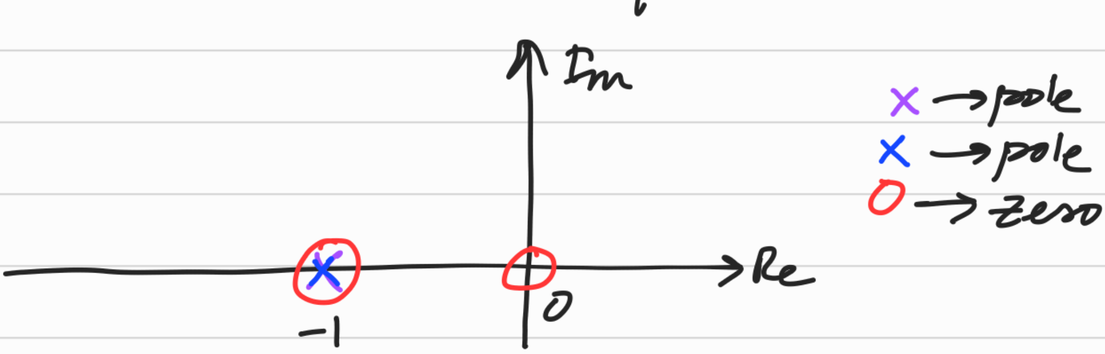
$$\boxed{\frac{Y_2(s)}{X(s)} = \frac{0.333s^2 + 0.333s - 6.831\times 10^{-18}}{s^2+2s+1}}$$

(b) pole-zero diagram :

for $\dfrac{Y_1(s)}{X(s)} = \dfrac{(s+1)}{(3s^2+6s+3)}$ → one zero @ -1
→ two poles @ $-1, -1$



for $\dfrac{Y_2(s)}{X(s)}$, we have zeros @ $0, -1$
poles @ $-1, -1$



(c) System BIBO Stable ?

→ since all poles of both transfer function are on strict LHP so the system is BIBO stable.

Pb#03: Unit step Response :

$$\dot{q} = \underbrace{\begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}}_{A} q + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{B} x \,, \quad y = \underbrace{\begin{bmatrix} -1 & 1 \end{bmatrix}}_{C} q$$

(a) compute $y(t)$, unit step output response, when the initial condition is zero.

Since, $Y(s) = H(s)X(s)$, where $X(s) = \frac{1}{s}$ (unit step)

and

$$H(s) = C(sI - A)^{-1}B$$

where,

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (s+1) & 3 \\ -1 & (s-2) \end{bmatrix}$$

and

$$(sI - A)^{-1} = \frac{1}{(s+1)(s-2)+3} \begin{bmatrix} (s-2) & -3 \\ 1 & (s+1) \end{bmatrix}$$

$$= \frac{1}{s^2 - 2s + s - 2 + 3} \begin{bmatrix} (s-2) & -3 \\ 1 & (s+1) \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 - s + 1} \begin{bmatrix} (s-2) & -3 \\ 1 & (s+1) \end{bmatrix}$$

And

$$(sI - A)^{-1}B = \frac{1}{s^2 - s + 1} \begin{bmatrix} -(s-2) \\ -1 \end{bmatrix}$$

So,

$$C(sI - A)^{-1}B = \frac{1}{s^2 - s + 1}(s - 2 - 1) = \boxed{\frac{(s-3)}{(s^2 - s + 1)} = H(s)}$$

So, $Y(s) = \frac{(s-3)}{s(s^2 - s + 1)}$, using ilaplace, we got

$$y(t) = 3e^{t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{9}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] - 3$$

$$y(t) = 3e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) - 3$$

(b) Simulated and calculated response comparision MATLAB.

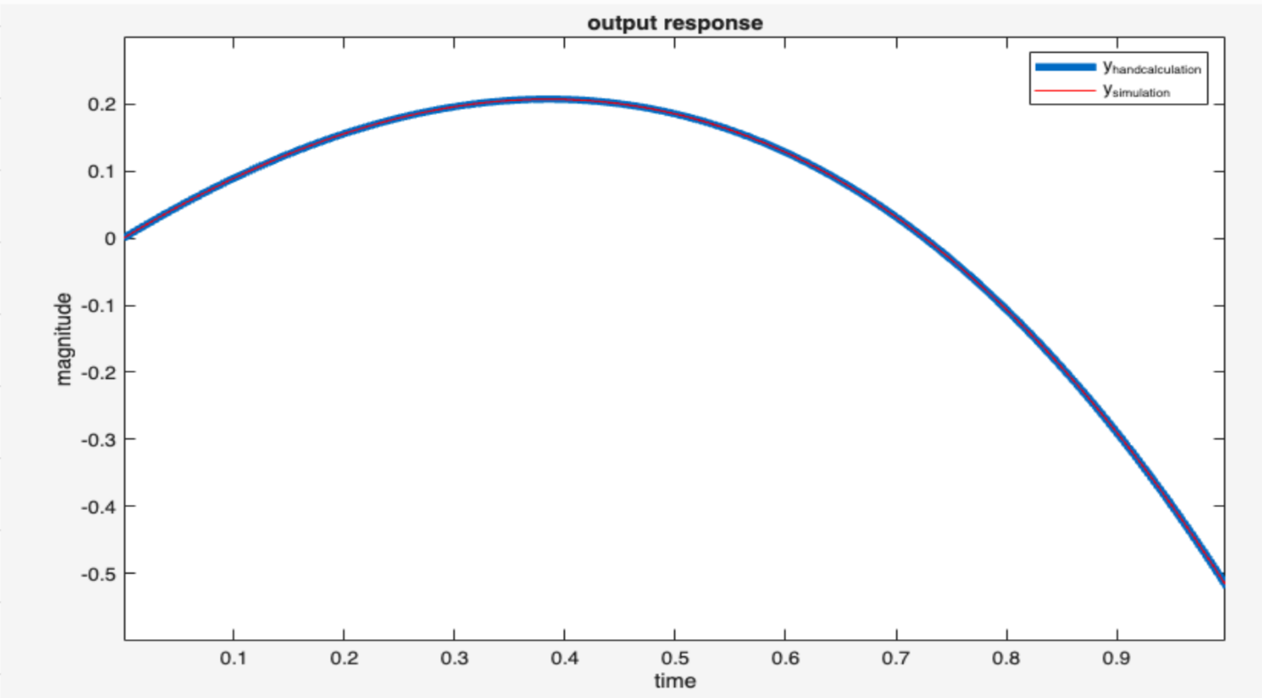
hw8_ece302.m × ss_to_tf.m ×

/Users/fatimaraza/Documents/MATLAB/hw8_ece302.m

```
clc
clear all
num=[0,1,-3];
den=[1,-1,1];
sys=tf(num,den)

t=0:0.01:1;
n=length(t);

for i=1:1:n
    y(i)= 3*exp(t(i)/2)*(cos((3^(1/2)*t(i))/2) - (3^(1/2)*sin((3^(1/2)*t(i))/2))/9) - 3;
end
y_simulation =step(sys,t);
plot(t,y,'LineWidth',4);
hold on
plot(t,y_simulation,'Red');
xlabel('time')
ylabel('magnitude')
legend('y_{handcalculation}','y_{simulation}')
title('output response')
```

Command Window

```
sys =

    s - 3
  -----------
  s^2 - s + 1

Continuous-time transfer function.
```



Pb#04: T.F to SS Model:

$$H(s) = \frac{(3s+1)}{(s^2+3s+7)}$$

(a) Distinct state space realizations.

since $$H(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

for $n=2$ here,

$$H(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}, \quad \text{where, } a_1 = 3,\ a_2 = 7,\ b_1 = 3,\ b_2 = 1$$

so,

(i) Controllable Cannonical form:

$$\dot{q} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} b_1 & b_2 \end{bmatrix} q$$

so,

$$\dot{q} = \overbrace{\begin{bmatrix} -3 & -7 \\ 1 & 0 \end{bmatrix}}^{A_1} q + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{b_1} x, \quad y = \overbrace{\begin{bmatrix} 3 & 1 \end{bmatrix}}^{c_1} q$$

(ii) Observable Cannonical form:

$$\dot{q} = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} q + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} q$$

so,

$$\dot{q} = \overbrace{\begin{bmatrix} -3 & 1 \\ -7 & 0 \end{bmatrix}}^{A_2} q + \overbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}^{b_2} x, \quad y = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{c_2} q$$

(b) Explicitly computing T.F for each realization.

(i) $H_1(s) = c_1(sI - A_1)^{-1} b_1$

so

$$(sI - A_1) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -7 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & 7 \\ -1 & s \end{bmatrix}$$

and

$$(sI - A_1)^{-1} = \frac{1}{s(s+3)+7} \begin{bmatrix} s & -7 \\ 1 & s+3 \end{bmatrix}$$

so,

$$(sI - A_1)^{-1} b_1 = \frac{1}{s^2+3s+7} \begin{bmatrix} s & -7 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c(sI - A_1)^{-1} b_1 = \frac{1}{s^2+3s+7} \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}$$

And so,

$$\boxed{H_1(s) = \frac{3s+1}{s^2+3s+7}}$$ verified!

(ii) $H_2(s) = c_2(sI - A_2)^{-1} b_1$

$$(sI - A_2) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 7 & s \end{bmatrix}$$

$$(sI - A_2)^{-1} b_2 = \frac{1}{s(s+3)+7} \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ -7 & s+3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$c_2(sI - A_2)^{-1} b_2 = \boxed{\frac{3s+1}{s^2+3s+7} = H_2(s)}$$ verified!