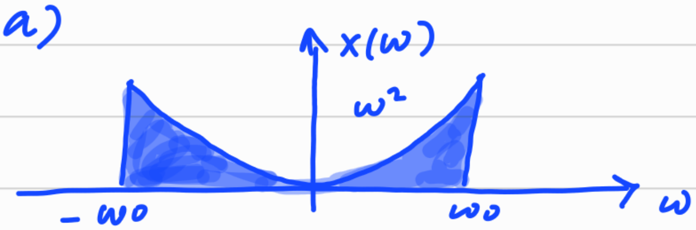
ECE 302 – HW#07 SOLUTIONS FALL 2025

Pb#01 (7.1–7): Inverse Fourier transforms of the Spectra: Fig P7.1–7

(a)



$$x(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\, d\omega =$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega t}\omega^2}{jt} - \frac{2}{j^2t^2}e^{j\omega t}\omega + \frac{2}{j^3t^3}e^{j\omega t}\right)\Bigg|_{-\omega_0}^{\omega_0}$$

$$x(t) = \frac{1}{2\pi}\underbrace{\frac{e^{j\omega t}}{j^3t^3}\left(j^2t^2\omega^2 - 2j\omega t + 2\right)}_{\text{let } A(\omega)}\Bigg|_{-\omega_0}^{\omega_0}$$

so, $x(t) = \frac{1}{2\pi}\left[A(\omega_0) - A(-\omega_0)\right]$

$$= \frac{1}{2\pi}\frac{1}{j^3t^3}\Big[e^{j\omega_0 t}(-\omega_0^2t^2 - 2j\omega_0 t + 2) - e^{-j\omega_0 t}(-\omega_0^2t^2 + 2j\omega_0 t + 2)\Big]$$

$$x(t) = \frac{1}{2\pi j^3t^3}\Big[(-\omega_0^2t^2+2)e^{j\omega_0 t} - 2j\omega_0 t e^{j\omega_0 t} - (-\omega_0^2t^2+2)e^{-j\omega_0 t} - 2j\omega_0 t e^{-j\omega_0 t}\Big]$$

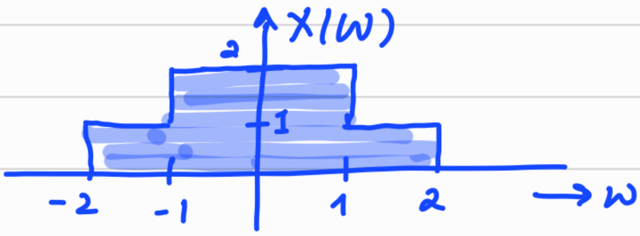
$$= \frac{1}{2\pi j^3t^3}\Big[(-\omega_0^2t^2+2)\overbrace{(e^{j\omega_0 t} - e^{-j\omega_0 t})}^{=2j\sin(\omega_0 t)} - 2j\omega_0 t\underbrace{(e^{j\omega_0 t} + e^{-j\omega_0 t})}_{=2\cos(\omega_0 t)}\Big]$$

so, $x(t) = \frac{1}{\pi t^3}\left[\frac{-(w_0^2t^2+2)}{\cancel{2}j^{2}}\left(\cancel{2j}\sin(w_0t)\right) - \frac{\cancel{2j}w_0t\left(2\cos(w_0t)\right)}{\cancel{2}j^{2}}\right]$

$$= \frac{1}{\pi t^3}\left[-\frac{(w_0^2t^2+2)\sin(w_0t)}{-1} - \frac{2w_0t\cos(w_0t)}{-1}\right]$$

$$\boxed{x(t) = \frac{(w_0^2t^2+2)\sin(w_0t) + 2w_0t\cos(w_0t)}{\pi t^3}}$$

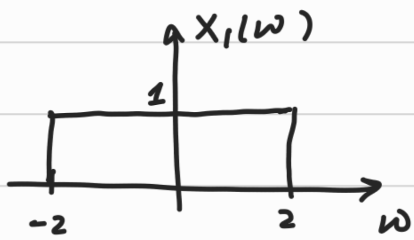
(b) $X(w)$ can be expressed as the sum of two $X_1(w)$ and $X_2(w)$



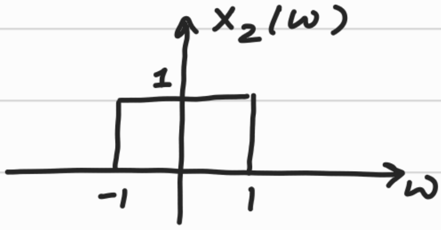
Therefore,

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2}\left[X_1(w) + X_2(w)\right]e^{jwt}\,dw$$

So,



and



so,

$$x(t) = \frac{1}{2\pi}\left\{\int_{-2}^{2} e^{jwt}\,dw + \int_{-1}^{1} e^{jwt}\,dw\right\}$$

$$= \frac{1}{\pi}\left(\frac{e^{j2t} - e^{-j2t}}{2jt} + \frac{e^{jt} - e^{-jt}}{2jt}\right\}$$

$$\boxed{x(t) = \frac{1}{\pi t}\left\{\sin(2t) + \sin(t)\right\}}$$

Pb#02 (7.2-3): show inverse Fourier transform of $\text{rect}\left(\frac{\omega-10}{2\pi}\right)$ is $\text{sinc}(\pi t)e^{j10t}$

$$x(t) = \frac{1}{2\pi}\int_{10-\pi}^{10+\pi} e^{j\omega t}\,d\omega = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Big|_{10-\pi}^{10+\pi}$$

$$x(t) = \frac{1}{j2\pi t}\left(e^{j(10+\pi)t} - e^{j(10-\pi)t}\right)$$

$$= \frac{e^{j10t}}{\pi t}\left(\frac{e^{j\pi t} - e^{-j\pi t}}{j2}\right) = \frac{e^{j10t}}{\pi t}\left(\sin(\pi t)\right)$$

and so,

$$\boxed{x(t) = e^{j10t}\,\text{sinc}(\pi t)}$$

Since, $\frac{\sin x}{x} = \text{sinc}(x)$

Pb#03 (7.3-2): Fourier transform $Y(\omega)$ in terms of $X(\omega)$

(a) $y(t) = \frac{1}{5}x(-2t+3)$

start: $x(t) \Longleftrightarrow X(\omega)$

time scaling: $x(-2t) \Longleftrightarrow \frac{1}{2}X\left(\frac{-\omega}{2}\right)$

time shifting: $x\left(-2\left(t-\frac{3}{2}\right)\right) \Longleftrightarrow \frac{e^{-j3\omega/2}}{2}X\left(\frac{-\omega}{2}\right)$

amplitude scaling: $\underbrace{\frac{1}{5}x\left(-2\left(t-\frac{3}{2}\right)\right)}_{=y(t)} \Longleftrightarrow \underbrace{\frac{e^{-j3\omega/2}}{2\times5}X\left(-\frac{\omega}{2}\right)}_{Y(\omega)}$

so,

$$\boxed{Y(\omega) = \frac{1}{10}e^{-j3\omega/2}X\left(-\frac{\omega}{2}\right)}$$

(b) $y(t) = e^{j2t} x^*(-3t-6)$

start: $x(t) \Longleftrightarrow X(\omega)$

time scaling: $x(-3t) \Longleftrightarrow \frac{1}{3} X\left(\frac{-\omega}{3}\right)$

conjugation: $x^*(-3t) \Longleftrightarrow \frac{1}{3} X^*\left(\frac{\omega}{3}\right)$

time shifting: $x^*(-3(t+2)) \Longleftrightarrow \frac{1}{3} e^{j2\omega} X^*\left(\frac{\omega}{3}\right)$

frequency shifting: $\underbrace{e^{j2t} x^*(-3(t+2))}_{y(t)} \Leftrightarrow \underbrace{\frac{e^{j2(\omega-2)}}{3} X^*\left(\frac{\omega-2}{3}\right)}_{Y(\omega)}$

So,

$$\boxed{Y(\omega) = \frac{1}{3} e^{j2(\omega-2)} X^*\left(\frac{\omega-2}{3}\right)}$$

Pb#04 (7.4-4): periodic delta train $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-\pi n)$

$x(t) \longrightarrow \boxed{\text{LTIC}} \longrightarrow y(t) = x(t) * h(t)$

where,

Impulse response: $h(t) = \sin(3t)\, \text{sinc}^2(t)$

Note: Some textbook have $h(t) = \sin(3t)\, \text{sinc}^2\left(\frac{t}{\pi}\right)$ for pb#7.4-4. Both are acceptable.

(a) $\omega_0 = ?$ (fundamental radian frequency of $x(t)$)

$x(t)$ has period $T_0 = \pi$ gives $\omega_0 = \frac{2\pi}{T_0} \Rightarrow \boxed{\omega_0 = 2}$

(b) Fourier transform $X(\omega)$:

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \quad , \quad \text{use } T_0 = \pi, \ \omega_0 = 2$$

$$\boxed{X(\omega) = 2 \sum_{n=-\infty}^{\infty} \delta(\omega - 2n)} \quad \text{Another delta train}$$

(c) Sketch $|H(\omega)|$ over $-10\pi \le \omega \le 10\pi$

For given $h(t) = \sin(3t)\,\mathrm{sinc}^2(t)$

from table 7.1

$$\sin(3t) \Longleftrightarrow j\pi\,[\delta(\omega+3) - \delta(\omega-3)]$$

and

$$\mathrm{sinc}^2(t) \Longleftrightarrow \pi\,\Delta\left(\frac{\omega}{4}\right)$$

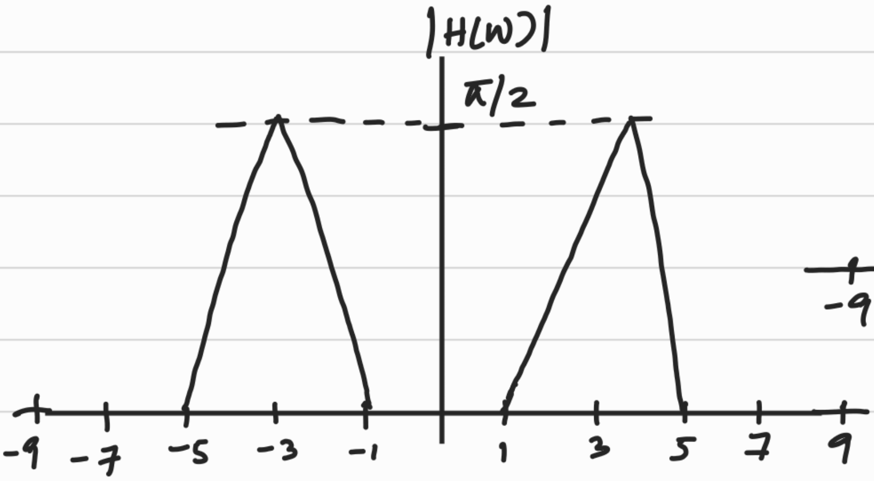
using the frequency convolution property,

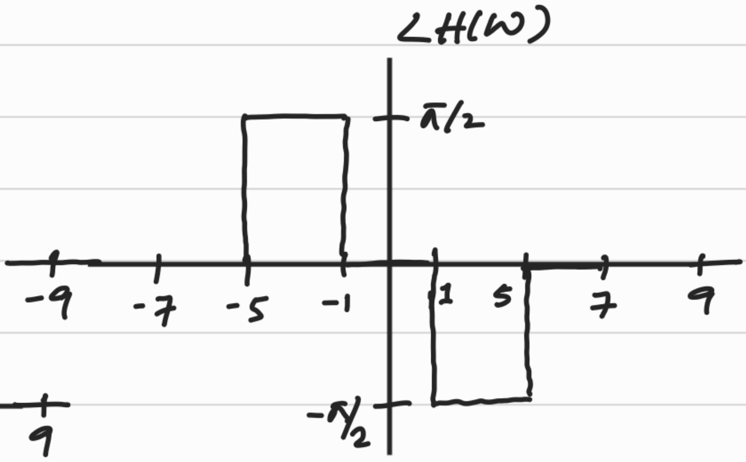
$$h(t) = \sin(3t)\,\mathrm{sinc}^2(t) \Longleftrightarrow$$

$$\frac{1}{2\pi}[j\pi\,\delta(\omega+3) - j\pi\,\delta(\omega-3)] * \pi\,\Delta\left(\frac{\omega}{4}\right) = H(\omega)$$

Thus,

$$H(\omega) = \frac{j\pi}{2}\,\Delta\left(\frac{\omega+3}{4}\right) - \frac{j\pi}{2}\,\Delta\left(\frac{\omega-3}{4}\right)$$





(d) is System h(t) distortionless?

Distortionless systems have constant magnitude response and linear phase response. from part (c), we can observe the magnitude response is not constant. Thus,

since $|H(\omega)|$ is not constant, the System is Not distortionless.

(e) Determine $y(t) = ?$

The input $x(t)$ is comprised of component frequencies $\omega = 0, \pm 2, \pm 4, \pm 6, \ldots$ rad/sec. Inspection of Sketches from part (c) confirms that only the components at $\omega = \pm 2$ and $\omega = \pm 4$ will pass through the system (with gains $\frac{\pi}{4}$ and phases $\pm\frac{\pi}{2}$).

In the frequency domain, the output is:

$$Y(\omega) = X(\omega)H(\omega) = 2\delta(\omega+4)\frac{j\pi}{4} + 2\delta(\omega+2)\frac{j\pi}{4}$$

$$+ 2\delta(\omega-2)\frac{(-j\pi)}{4} + 2\delta(\omega-4)\frac{(-j\pi)}{4}$$

So, inverting, we obtain,

$$y(t) = \frac{1}{2\pi}\left(\frac{j\pi}{2}e^{-j4t} + \frac{j\pi}{2}e^{-j2t} - \frac{j\pi}{2}e^{j2t} - \frac{j\pi}{2}e^{j4t}\right)$$

$$= \frac{1}{2}\left(\frac{e^{j4t} - e^{-j4t}}{2j} + \frac{e^{j2t} - e^{-j2t}}{2j}\right)$$

$$\boxed{y(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}\sin(4t)}$$

Pb#05: Laplace v/s Fourier Transform

Suppose: $x(t) = 5e^{2t}u(-t) + 2u(t-1)$

(a) Two-sided Laplace transform:

note: $u(-t) = \begin{cases} 1, & t \leq 0 \\ 0, & t \geq 0 \end{cases}$

and

$$u(t-1) = \begin{cases} 1, & t \geq 1 \\ 0, & t \leq 1 \end{cases}$$

So,

$$x(s) = \int_{-\infty}^{0} 5e^{2t}e^{-st}\,dt + \int_{1}^{\infty} 2e^{-st}\,dt$$

$$x(s) = 5\int_{-\infty}^{0} e^{(2-s)t}\,dt + 2\int_{1}^{\infty} e^{-st}\,dt$$

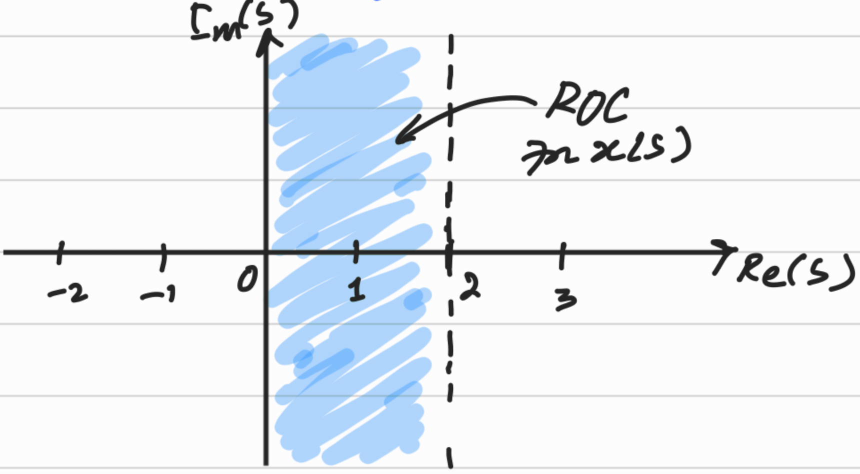
$$x(s) = 5\,\frac{e^{(2-s)t}}{(2-s)}\Bigg|_{-\infty}^{0} + 2\,\frac{e^{-st}}{-s}\Bigg|_{1}^{\infty}$$

$$= \frac{5e^{0}}{(2-s)} - \underbrace{\lim_{t\to-\infty}\frac{5e^{(2-s)t}}{(2-s)}}_{\substack{=0 \\ \text{as } Re(2-s)>0 \\ Re(s)<2}} + \underbrace{\lim_{t\to\infty}\frac{2e^{-st}}{-s}}_{\substack{=0 \\ Re(s)>0}} - \left(-\frac{2e^{-s}}{-s}\right)$$

So,

$$\boxed{x(s) = \frac{5}{2-s} + \frac{2e^{-s}}{s}}$$ for $0 < Re(s) < 2$

(b) Region of convergence for $x(s)$ from part (a)



as,

$$0 < Re(s) < 2$$

(c) Fourier transform $x(t)$ when $s = j\omega$

Fourier transform is <u>not</u> equal to the Laplace transform with $s = j\omega$ since $s = j\omega = 0 + j\omega$ give $Re(s) = 0$ that is not in ROC i.e $0 < Re(s) < 2$
(or say $j\omega$ axis is not in ROC)

(d) Fourier transform of $x(t)$

$$x(t) = 5e^{2t}u(-t) + 2u(t-1)$$

<u>Table 7.1:</u> (2) $e^{at}u(-t) \Longleftrightarrow \dfrac{1}{a - j\omega}$

So, $5e^{2t}u(-t) \Longleftrightarrow \dfrac{5}{2 - j\omega}$

and Also note:

$$u(t) \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

time shift: $u(t - t_0) \Longleftrightarrow e^{-j\omega t_0}\left[\pi\delta(\omega) + \dfrac{1}{j\omega}\right]$

So,

$$2u(t-1) \Longleftrightarrow 2e^{-j\omega}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

So,

Fourier transform of $x(t)$ is:

$$X(w) = \frac{5}{2-jw} + 2e^{-jw}\left[\pi\delta(w) + \frac{1}{jw}\right]$$

or

$$\boxed{X(w) = \frac{5}{2-jw} + \frac{2e^{-jw}}{jw} + 2\pi e^{-jw}\delta(w)}$$

note: $X(w)$ (Fourier transform of $x(t)$) is Not equal to $X(s)$ (Laplace transform of $x(t)$ from part (a)) for $s=jw$